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A global optimization algorithm for generalized semi-infinite, continuous minimax with coupled constraints and bi-level problems

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Abstract We propose an algorithm for the global optimization of three problem classes: generalized semi-infinite, continuous coupled minimax and bi-level problems. We make no convexity assumptions. For each problem class, we construct an oracle that decides whether a given objective value is achievable or not. If a given value is achievable, the oracle returns a point with a value better than or equal to the target. A binary search is then performed until the global optimum is obtained with the desired accuracy. This is achieved by solving a series of appropriate finite minimax and min-max-min problems to global optimality. We use Laplace's smoothing technique and a simulated annealing approach for the solution of these problems. We present computational examples for all three problem classes.

Keywords Generalized semi-infinite · Minimax · Bi-level · Globaloptimization · Min-max-min

1 Introduction

In this paper we provide a method for the global optimization of three related problems. The first problem is the Generalized Semi Infinite Problem (GENSI):

$$\min_x f(x) \tag{P_1}$$

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subject to

$$x \in M = \{x \in A \subseteq \mathbb{R}^n | g(x, y) \leq 0, \quad \forall y \in Y(x)\}$$

with

$$Y(x) = \{y \in B \subseteq \mathbb{R}^m | v_i(x, y) \leq 0, i \in I\}$$

and I being a finite index set. Applications of GENSI include lapidary cutting problems [1], reverse Chebyshev approximation [2] and minimal time control [3]. For a survey on GENSI the reader is referred to [4]. If Y does not depend on x then we have a standard semi-infinite problem. For a survey see [5]. A global optimization algorithm for standard semi-infinite programming is presented in [6]. To the best of our knowledge, no global optimization algorithm for the generalized semi infinite problem has been presented so far.

The second problem we consider is the continuous minimax problem with coupled constraints (MMC)

$$\min_{x \in A \subseteq \mathbb{R}^n} \max_{y \in B \subseteq \mathbb{R}^m} f(x, y) \tag{P_2}$$

subject to

$$g_i(x, y) \leq 0, \quad i \in I,$$

where I is a finite set.

MMC's arise in multiple disciplines, including engineering [7,8], finance [9], farm planning [10], machine learning [11] and location problems [12]. In general they are used in decision under uncertainty to compute the worst case outcome when the decision is taken after the uncertainty has been resolved.

There are many local optimization algorithms in the literature that deal with the unconstrained continuous minimax problem [13,9] and the continuous minimax problem with decoupled constraints [14–16], i.e., a problem of the type

$$\min_{x \in X \subseteq R^n} \max_{y \in Y \subseteq R^m} f(x, y).$$

In [17] a global optimization algorithm is given for the decoupled constraints case where f, X, Y are described by polynomials.

However, few contributions are available that address the case of coupled constraints. Under convexity and concavity assumptions a relaxation procedure has been presented in [18], and an interior point algorithm in [15]. A parametric approach for the linear case is presented in [7]. In [19], the constraint $max_y g(x, y) \leq 0$ is considered. This constraint is easier to treat because firstly it restricts only x, and secondly it does not depend on the individual y choices. In essence it is a constraint on x which is hard to calculate, but not a coupled constraint. Royset et al. [20] present an algorithm that solves the coupled constraint problem for general functions to local optimality. They use an exact penalty function and transform the problem into a min-max-min problem.

The third problem, is the bi-level programming problem (BLP)

$$\min_{x,y} F(x, y), \tag{P_3}$$

subject to

$$G(x, y) \leq 0,$$

$$x \in X,$$

$$y \in \arg\min\{f(x, y) : g(x, y) \leq 0, \quad \forall y \in Y\}.$$

BLP can be interpreted as a leader follower game. The leader chooses a vector x, and the follower responds with a vector y optimizing his own objective (given x). The leader seeks to find the optimum x, taking into consideration the impact that his choice has on the follower. The formulation implies that whenever the follower has more than one optimal solution, the leader can choose the one that optimizes his own objective.

When the inner problem is convex, the most common approach is to substitute the inner problem with its Karush–Kuhn–Tucker (KKT) conditions to obtain an equivalent single-level problem. The transformed problem fails to be convex even in the linear case because of the complementary conditions, and therefore global optimization algorithms are needed to obtain the global optimum. A global optimization algorithm for the linear and quadratic case is presented in [21]. Recently, techniques from parametric global optimization [22] have been used for the same class of problems [23]. If the inner problem is non-convex, an extra difficulty is that a point might satisfy the KKT conditions while failing to be the global optimum of the inner problem.The first valid algorithm for the general non-convex inner problem case was presented recently in [24].

The three problems are closely related. The coupled minimax problem is a special case of the bi-level programming problem with $G = g$ and $f = -F$. Thus, the algorithm in [24] can also handle it. In [25] it is observed that if $Y(x)$ is nonempty for all x, (P_1) can be formulated as the bilevel program

$$\min_x f(x)$$

subject to

$$g(x, y_0) \leq 0,$$

$$y_0 \in \arg\min_y\{-g(x, y) : y \in Y(x)\}.$$

The opposite is also possible. That is, the bi-level problem (P_3) can be formulated as the GENSI problem

$$\min_{x, y_f} F(x, y_f)$$

subject to

$$G(x, y_f) \leq 0,$$

$$g(x, y_f) \leq 0,$$

$$x \in X$$

$$f(x, y_f) - f(x, y) \leq 0 \quad \forall y \in Y(x)$$

with

$$Y(x) = \{y \in Y : g(x, y) \leq 0\}.$$

Since we made no assumptions, it is the GENSI problem that is the most general one.

Table 1 Initial lower bounds

Problem class	Initial LB
GENSI	$\min_{x \in A} f(x)$
MMC	$\min_{x \in A, y \in B}\{f(x, y) : g(x, y) \leq 0\}$
BLP	$\min_{x \in X, y \in Y}\{F(x, y) : G(x, y) \leq 0\}$

In all three cases, instead of solving the problem directly, we ask whether we can achieve a target value f_0. If we can provide a definite answer to this question for an arbitrary f_0, then we can do a binary search in the space of objective values to identify a point that is nearly globally optimal. Note that the feasible region of a GENSI problem may not be closed, and thus the problem may not have a global optimum [25].

For a given target value we construct an unconstrained min-max-min optimization problem with an optimal value smaller or equal to 0 if f_o is achievable and greater than zero otherwise. The outer minimization and the maximization of the problem are over continuous variables while the inner minimization is discrete.

To solve the problem we discretize the maximization variables and dynamically add points to the discrete approximation as described in the next Section. The min-max-min problem need not be solved to optimality: as soon as the bounds generated by the discrete approximations allow to decide whether the optimal value is larger than 0 or not, the process can be terminated.

In order to start the binary search we need initial upper and lower bounds. For an initial upper bound on f_o we only need an evaluation of the objective function at a feasible point. Computing lower bounds is not hard either, as long as we do not demand them to be particularly tight. One possible way to obtain initial lower bounds is summarized in Table 1. Note however, that for all practical problems this may not be necessary. Any "good" guess will suffice since we do not require the lower bound to be particularly tight. The oracle can decide much faster when the target value is far from the optimum value. Thus initial tight bounds are less important than it may appear.

The remaining of the paper is organized as follows. In Sects. 2, 3 and 4 we construct oracles for the GENSI, MMC and BLP problems respectively. In Sect. 5 we discuss the global optimization algorithm used for the solution of the finite minimax and min-max-min subproblems. In Sect. 6 we provide numerical results and in Sect. 7 we conclude.

2 GENSI

Given a target value f_0 we are looking for an x such that

$$f(x) - f_0 \leq 0 \tag{1}$$

and

$$\min\{g(x, y), \min_i -v_i(x, y) + \epsilon\} \leq 0 \quad \forall y \in B \tag{2}$$

where $\epsilon > 0$ is an infeasibility tolerance. Equation 2 ensures x is feasible since it can be read as

$$\forall y, \; v_i(x, y) < \epsilon \quad \forall i \Rightarrow g(x, y) \leq 0.$$

We formulate the unconstrained min-max-min problem

$$\min_{x\in A}\max_{y\in B}\max\{f(x)-f_0,\min\{g(x,y),\min_i -v_i(x,y)+\epsilon\}\}. \tag{3}$$

Proposition 2.1 *If the optimal value of (3) is less than or equal to 0 then f_o is achievable for (P_1).*

Proof If x_o is the optimal solution to (3) and the optimal value of (3) is less than or equal to 0, then we have

$$f(x_o)\le f_o$$

and

$$\forall y,\ v_i(x_0,y)\le 0\quad \forall i \Rightarrow v_i(x_0,y)<\epsilon\quad \forall i\Rightarrow g(x_0,y)<0.$$

Therefore x_o is a feasible solution of (P_1) and achieves a value better than or equal to f_o. □

Proposition 2.2 *If for every $\epsilon>0$ the optimal value of (3) is larger than 0 then f_o is not achievable for (P_1).*

Proof Assume that f_o is achievable, that is, there exists a feasible solution of (P_1) x_1, with $f(x_1)\le f_0$. Since x_1 is feasible, we have

$$\forall y, \exists i\ v_i(x_1,y)=\alpha>0\ \text{ or }\ g(x_1,y)\le 0.$$

Therefore

$$\forall y \min\{g(x,y),\min_i -v_i(x,y)+\alpha\}\le 0$$

and

$$\max_{y\in B}\max\{f(x_1)-f_0,\min\{g(x_1,y),\min_i -v_i(x_1,y)+\alpha\}\}\le 0$$

which contradicts the hypothesis for $\epsilon<\alpha$ and thus f_o is not achievable. □

A short remark is in place to justify the use of ϵ. If we solve the problem

$$\min_{x\in A}\max_{y\in B}\max\{f(x)-f_0,\min\{g(x,y),\min_i -v_i(x,y)\}\}$$

instead and the optimal value is 0, it is not possible to tell whether f_o is achievable or not. For example, if at the optimal solution (x_0,y_0) there is a $j\in I$ with

$$v_j(x_0,y_0)=0,\quad v_i(x_0,y_0)<0,\quad \forall i\ne j,$$

$$g(x_0,y_0)>0,$$

$$f(x_0)\le f_0,$$

then the optimal solution will be 0 but the point (x_0,y_0) is not feasible. If, on the other hand, we have

$$v_i(x_0,y_0)<0,\quad \forall i,$$

$$g(x_0,y_0)=0,$$

$$f(x_0) \leq f_0,$$

then (x_0, y_0) is feasible and we know f_o is achievable. This is not just a technicality but can happen in practical problems.

We use a two phase procedure for the solution of (3) as in [26]. At every iteration, the algorithm attains a finite set $Y_k \subset B$ and solves the problem

$$\min_{x \in A} \max_{y \in Y_k} \max\{f(x) - f_0, \min\{g(x, y), \min_i -v_i(x, y) + \epsilon\}\}. \tag{4}$$

If at an iteration the optimal solution of (4) has a positive objective F_k^l. the oracle states that f_o is not achievable. This is justified because the optimal value of (4) defines a lower bound on the optimal value of (3). Otherwise if it returns the point x_k the problem

$$\max_{y \in B} \min\{g(x_k, y), \min_i -v_i(x_k, y) + \epsilon\}\} \tag{5}$$

is solved.

If (5) returns a value F_k^u less than or equal to 0, the oracle announces that f_o is achievable and x_k guarantees an objective better than or equal to f_o. This is because F_k^u is an upper bound for the optimal value of (3). If, on the other hand, (5) returns a value larger than 0, the solution y_{k+1} is added to Y_k and phase one is run with the new set Y_{k+1}. It is shown in [26] that solutions of the problems (4) generate subsequences of points converging to a solution of problem (3) and thus the procedure terminates.

Computational experiments have revealed that the algorithm performs better if we maintain a number of maximizers in Y_k, when we update f_o.

3 Minimax problems with coupled constraints

Consider the problem,

$$\min_{x \in A \subset \mathbb{R}^n} \max_{y \in B \subset \mathbb{R}^m} f(x, y)$$

subject to

$$g_i(x, y) \leq 0 \quad \forall i \in I,$$

where I is a finite set.

In the same vein as in Sect. 2 we ask whether a given value f_o is achievable. That is, we are looking for an x with

$$\exists y_f, \; g_i(x, y_f) \leq 0 \quad \forall i \in I, \tag{6}$$

$$\forall y \in B, \min\{\min_i -g_i(x, y) + \epsilon, f(x, y) - f_0\} \leq 0. \tag{7}$$

Equation 6 guarantees that x is feasible. Equation 7 ensures that for this x, every feasible y gives an objective value better than or equal to the target objective.

In order to find such an x, we solve the problem

$$\min_{x \in A, y_f \in B} \max\{\max_{i \in I} g_i(x, y_f), \max_{y \in B} \min\{\min_i -g_i(x, y) + \epsilon, f(x, y) - f_0\}.$$

We solve this problem by the same two phase algorithm. It is easy to see that the discrete approximation solved in phase 1,

$$\min_{x\in A, y_f\in B} \max\{\max_{i\in I} g_i(x, y_f), \max_{y\in Y_k} \min\{\min_i -g_i(x, y) + \epsilon, f(x, y) - f_0\}$$

is a finite min-max-min problem. The corresponding phase 2 problem is the finite minimax problem

$$\max_{y\in B} \min\{\min_i -g_i(x, y) + \epsilon, f(x, y) - f_0\}.$$

The algorithm terminates affirmatively whenever phase 1 returns a positive objective value and negatively whenever phase 2 returns a negative one or zero.

4 Bi-level problems

In the Bi-level problem, given the target f_o we are looking for an x, y_f with

$$F(x, y_f) - f_o \leq 0 \tag{8}$$

$$G_i(x, y_f) \leq 0 \tag{9}$$

$$g_j(x, y_f) \leq 0 \tag{10}$$

$$\forall y \in Y, \min\{\min_j -g_i(x, y) + \epsilon, f(x, y_f) - f(x, y)\} \leq 0. \tag{11}$$

Equation 11 ensures that y_o is a solution to the follower's problem given x, and thus that (x, y_f) is feasible.

The min-max-min problem we need to solve is

$$\begin{aligned}\min_{x,y_f} \max\{&F(x, y_f) - f_o,\\ &\max_i G_i(x, y_f),\\ &\max_j g_j(x, y_f),\\ &\max_{y\in Y} \min\{\min_j -g_i(x, y) + \epsilon, f(x, y_f) - f(x, y)\}\}.\end{aligned}$$

The discrete approximation is

$$\begin{aligned}\min_{x,y_f} \max\{&F(x, y_f) - f_o,\\ &\max_i G_i(x, y_f),\\ &\max_j g_j(x, y_f),\\ &\max_{y\in Y_k} \min\{\min_j -g_i(x, y) + \epsilon, f(x, y_f) - f(x, y)\}\},\end{aligned}$$

and the corresponding phase 2 problem is

$$\max_{y\in Y} \min\{\min_j -g_i(x_k, y) + \epsilon, f(x_k, y_{f,k}) - f(x_k, y)\}.$$

In addition to the termination criterion used for GENSI problems, we terminate affirmatively whenever phase 2 when feeded by phase 1 with x_k, computes a y_k with

$$F(x_k, y_k) \leq f_o \tag{12}$$

and

$$G_i(x_k, y_k) \leq 0 \quad \forall i. \tag{13}$$

This is to avoid waiting for y_f to converge to the actual optimum of the inner problem, which is not always necessary to correctly announce that f_o is achievable. From Eq. 13 and the fact that y_k solves the inner problem at x_k we know that (x_k, y_k) is feasible and if it achieves a value better than f_o as Eq. 12 states, f_o can be declared achievable.

5 Global optimization of subproblems

The algorithm described in Sect. 1 assumes that all subproblems are solved to global optimality. This is less important for problem (4) if it returns a value less than 0, and for problem (5) if it returns a value larger than 0, since the algorithm does not terminate and will have another chance for a better search. Whenever the oracle returns and announces an answer however, it is vital that the last optimization problem is solved to global optimality.

For both problems we use a smoothing technique based on the Laplace method which was introduced as a tool for minimax and min-max-min problems by Polak [27,28]. In the min-max-min problem, we apply the smoothing technique twice to reduce the problem to the minimization of a smooth function.

The maximin problem

$$\max_x \min_i f_i(x)$$

is approximated by

$$\max_x \Phi_\lambda(x) = -\frac{1}{\lambda} \ln\left(\sum_i e^{-\lambda f_i(x)}\right),$$

and the min-max-min problem

$$\min_x \max_i \min_j f_{i,j}(x)$$

is approximated by

$$\min_x \Phi_\lambda(x) = \frac{1}{\lambda} \ln\left(\sum_i \frac{1}{\sum_j e^{-\lambda f_{i,j}(x)}}\right),$$

where $\lambda > 0$ is the smoothing parameter [29]. We use the stochastic global optimization algorithm described in [30] for the solution of the problems:

The stochastic search method follows the trajectory of the Langevin equation

$$dX(t) = -\nabla\Phi_\lambda(X(t))dt + T(t)dB(t), \tag{14}$$

where $T(t)$ is the "cooling" schedule and $B(t)$ the standard Brownian motion. It is shown in [30] that the solution of (14) $X(t)$ converges to the global optima of $\Phi_\lambda(X(t)$ as t goes to infinity.

The quality of the solution of the overall algorithm depends on the quality of the global optimizer used for the subproblems. Since we are using a stochastic algorithm for the subproblems, even though our analysis was exact for the higher level algorithm, the overall construction remains stochastic. In principle, it is possible to solve the problems using a deterministic approach.

In order to solve the finite min-max-min problem

$$\min_x \max_i \min_j f_{i,j}(x),$$

one can use a branch and bound approach, where the lower bounds are calculated by

$$\max_i \min_j \min_x f_{i,j}(x)$$

and for upper bounds we use function evaluations. For the solution of the problems

$$\min_x f_{i,j}(x)$$

there are many deterministic global optimization algorithms in the literature, e.g. [31].

We note, however, that if one uses a deterministic method, it would be advantageous to combine it with a faster stochastic method. The deterministic one should be used whenever the stochastic one asserts than no feasible point with value better than 0 exists—less than 0 for (4) and larger than 0 for (5)—to check whether the stochastic's method assertion is indeed correct.

6 Numerical examples

Example 1 The first example is a continuous minimax problem with coupled constraints taken from [20].

$$\min_{x_1,x_2,x_3} \max_y 3(x_1 - y)^2 + (2 - y)(x_2)^2 + 5(x_3 + y)^2 + 2x_1 + 3x_2 - x_3 + e^{4y^2}$$

subject to

$$\frac{1}{4}\sin(x_1x_2) + y - \frac{1}{2} \leq 0$$

$$0 \leq y \leq 1.$$

In [20] the local minimizer

$$x = \{-0.0033, -1.0002, -0.3928\}$$

is obtained with an objective value 2.4100.

We run our algorithm with a starting objective window [−10,10]. The results of each iteration are given in Table 2. At the best point attained

$$x = \{-0.38623, -1.18495, -0.28284\}$$

the maximizer is $y = 0.38953674$ and the objective value 1.91399.

Table 2 Numerical results of Example 1

Target f_o	x_1	x_2	x_3	Achieved	New window
0				No	[0,10]
5	0.455073	−1.22052	−0.5095	Yes	[0.5]
2.5	−0.45986	−1.29386	−0.3026	Yes	[0,2.5]
1.25				No	[1.25,2.5]
1.875				No	[1.875,2.5]
2.1875	−0.45986	−1.29386	−0.3026	Yes	[1.875,2.1875]
2.03125	−0.45986	−1.29386	−0.3026	Yes	[1.875,2.03125]
1.95313	−0.45272	−1.23876	−0.25808	Yes	[1.875,1.95313]
1.91406	−0.38623	−1.18495	−0.28284	Yes	[1.875,1.91406]

Table 3 Numerical results of Example 2

Target f_o	x	y	Achieved	New window
0			No	[0,1]
0.5	-5.8×10^{-12}	0	Yes	[0,0.5]
0.25	-5.8×10^{-12}	0	Yes	[0,0.25]
0.125	-5.8×10^{-12}	0	Yes	[0,0.125]
0.0625	-5.8×10^{-12}	0	Yes	[0,0.0625]
0.03125	-5.8×10^{-12}	0	Yes	[0,0.03125]
0.0156	-5.8×10^{-12}	0	Yes	[0,0.0156]

Example 2 This example is a bilevel program taken from [25].

$$\min_{x,y} F(x, y) = y - x,$$

subject to

$$x \le 1,$$

$$y \in \arg\min\{-x - y : -x \cdot y \le 0,\ 0 \le y \le 2\}.$$

The feasible region is the open set

$$\{(x, 0) : x < 0\} \cup \{(x, 2) : 0 \le x \le 1\},$$

and the problem does not have a global optimal solution. A local minimizer is (x_0, y_0)=(1, 2) with $F(x_0, y_0) = 1$. We run our algorithm with an initial window $(-2, 2)$ and the results are given in Table 3. Our algorithm locates a feasible point (x_1, y_1) with $F(x_1, y_1) = 5.8 \times 10^{-12}$.

Example 3 Consider the bilevel program

$$\min_{x_1,x_2,x_3,y} 3(x_1 - y)^2 + (2 - y)(x_2)^2 + 5(x_3 + y)^2 + 2x_1 + 3x_2 - x_3 + e^{4y^2}$$

Table 4 Numerical results of Example 3

Target f_o	x_1	x_2	x_3	y	Achieved	New window
0	0.049755	−0.74818	0.099399	0	Yes	[−25,0]
−12.5					No	[−12.5,0]
−6.25					No	[−6.25,0]
−3.125					No	[−3.125,0]
−1.5625					No	[−1.5625,0]
−0.78125					No	[−0.78125,0]
−0.39063					No	[−0.39063,0]
−0.19531	−0.02853	−0.75103	0.101894	0.03577	Yes	[−0.39063,−0.19531]
−0.29297					No	[−0.29297,−0.19531]
−0.24414					No	[−0.24414,−0.19531]
−0.21973					No	[−0.21973,−0.19531]
−0.207518	−0.03585	−0.74956	0.075502	0.035777	Yes	[−0.21973,−0.207518]
−0.213621	−0.05162	−0.76646	0.053737	0.051561	Yes	[−0.21973,−0.213621]

subject to

$$x_0 + x_1 \leq 2,$$

$$y \in \arg\min \left\{ (x_1 + y)^2 + x_2 \cdot \cos(x_3 \cdot y) : 0 \leq y \leq 1, \ \frac{1}{4} \ \sin(x_1 x_2) + y - \frac{1}{2} \leq 0 \right\}$$

We run the algorithm with an initial window of (−25,25) and the results are given in Table 4.

Example 4 Consider the following reverse Chebyshev approximation problem. We want to approximate the function $\sin(y)$ with a second degree polynomial to achieve an error less than 0.2 for all $y \in [1 - r, 1 + r]$ while maximizing r. The Taylor expansion

$$\sin(1) + \cos(1)\ (y - 1) - \frac{\sin(1)}{2}(y - 1)^2$$

achieves $r = 1.175$.

The problem can be formulated as a GENSI problem

$$\min_{r,\alpha,\beta,\gamma} \ -r$$

subject to

$$|\sin(y) - \alpha y^2 - \beta y - \gamma| \leq 0.2, \quad \forall y \in Y(r)$$

with

$$Y(r) = [1 - r, 1 + r],$$

$$r > 0.$$

We start our algorithm with an initial window [−10,0] and the results are given in Table 5. The approximation we derive achieves $r = 2.0996$.

In the appendix we present the first few iterations of a bi-level example problem to illustrate our proposed algorithm.

Table 5 Numerical results of Example 4

Target f_o	α	β	γ	Achieved	New window
−5				No	[−5,0]
−2.5				No	[−2.5,0]
−1.25	−0.29193	1.0315	0.05829	Yes	[−1.25,−2.5]
−1.875	−0.27688	0.902387	0.131236	Yes	[−1.875,−2.5]
−2.1875				No	[−1.875,−2.1875]
−2.03125	−0.26377	0.842673	0.153396	Yes	[−2.03125,−2.1875]
−2.10938				No	[−2.03125,−2.10938]
−2.07031	−0.26317	0.835821	0.155965	Yes	[−2.07031,−2.10938]
−2.08984	−0.26157	0.832648	0.161875	Yes	[−2.08984,−2.10938]
−2.09961	−0.25958	0.824561	0.160268	Yes	[−2.09961,−2.10938]

7 Concluding remarks

We have presented an algorithm for the global optimization of non-convex generalized semi-infinite, minimax with coupled constraints, and bi-level problems. To the best of our knowledge, in the generalized semi-infinite case, it is the first algorithm that can solve non-convex problems to global optimality. Until the recent work in [24], the global optimization of general non-convex bi-level problems and minimax with coupled constraints were also open problems. In our implementation we used a stochastic algorithm for the global optimization of the subproblems, but in principle a deterministic approach can be used.

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Appendix

To illustrate the algorithm we follow step by step the first few iterations of the following bi-level problem

$$\min_{x,y} F(x, y) = x \cdot y \tag{15}$$

subject to

$$-10 \leq x \leq 1$$

$$y \in \arg\min y^4 + 2.7y^3 + 0.3y^2 - 1.4x \cdot y$$

$$y \textbf{ free}.$$

In Phase one we solve problems of the form

$$\min_{x,y_f} \max\{x \cdot y - f_o, \max_{y_i \in Y_k} f(x, y_f) - f(x, y_i)\}$$

with

$$f(y) = y^4 + 2.7y^3 + 0.3y^2 - 1.4x \cdot y).$$

In Phase 2 we solve

$$\max_y f(x, y_f) - f(x, y).$$

We denote with $PA = (x, y_f, obj)$ the results of phase one and with $PB = (y, obj)$ the results of phase two.

We start with an initial window [−4,0].

Iterations

- We start with $Y = \{0\}$ and solve

$$\min_{x, y_f} \max\{x \cdot y + 2, f(x, y_f) - f(x, 0)\}.$$

We obtain $PA = (1, -12.5388, -10.5388)$. We solve

$$\max_y f(x, -12.5388) - f(x, y),$$

and we obtain $PB = (-1.83997, 19462)$. We set

$$Y_k = \{0, -1.83997\}.$$

We solve

$$\min_{x, y_f} \max\{x \cdot y + 2, f(x, y_f) - f(x, 0), f(x, y_f) - f(x, -1.83997)\}.$$

and we obtain $PA = (1, -1.93924, \mathbf{0.06075 > 0})$ and $f_o = -2$ is not achievable. Note that only the last problem was solved to global optimality.

- We solve the phase 1 problem and we obtain $PA = (0.537175, -1.89459, -0.01768)$. We solve

$$\max_y f(0.537175, -1.89459) - f(0.537175, y)$$

and we obtain $PB = (-1.89333, -1.039 \times 10^{-5})$.We set

$$Y_k = \{0, -1.83997, -1.89333\}.$$

We have

$$F(0.537175, -1.89459) = -1.017 < f_0$$

and $f_0 = -1$ is achievable.

- We solve the phase 1 problem and we obtain $PA = (0.806226, -1.86449, -0.06319)$. We solve

$$\max_y f(0.806226, -1.86449) - f(0.806226, y)$$

and we obtain $PB = (-1.86321, -9.84 \times 10^{-6})$. We set

$$Y_k = \{0, -1.83997, -1.89333, -1.86321\}.$$

We have

$$F(0.806226, -1.86449) < f_0$$

and $f_0 = -1.5$ is achievable.

- We solve the phase 1 problem and we obtain $PA = (0.947709, -1.84685, -0.06024)$. We solve

$$\max_y f(0.947709, -1.84685) - f(0.947709, y)$$

and we obtain $PB = (-1.84648, -8.1 \times 10^{-7})$. We set

$$Y_k = \{0, -1.83997, -1.89333, -1.86321, -1.84685\}.$$

We solve the phase 1 problem and we obtain $PA = (0.974548, -1.84322, -6.1 \times 10^{-5})$. We solve

$$\max_y f(0.974548, -1.84322) - f(0.974548, y)$$

and we obtain $PB = (-1.84323, -2.2 \times 10^{-10})$. We set

$$Y_k = \{0, -1.83997, -1.89333, -1.86321, -1.84685, -1.84323\}.$$

We have

$$F(0.97454, -1.84323) < f_0$$

and $f_0 = -1.75$ is achievable.
- We solve the phase 1 problem and we obtain $PA = (1, -1.86983, \mathbf{0.605} > \mathbf{0})$. The value $f_0 = -1.875$ is not achievable.
- We solve the phase 1 problem and we obtain $PA = (0.983875, -1.84222, -2.5 \times 10^{-5})$. We solve

$$\max_y f(0.983875, -1.84222) - f(0.983875, y)$$

and we obtain $PB = (-1.84209, -9.3 \times 10^{-8})$. We set

$$Y_k = \{0, -1.83997, -1.89333, -1.86321, -1.84685, -1.84323, -1.84209\}.$$

We solve the phase 1 problem and we obtain $PA = (0.992533, -1.84104, -6.4 \times 10^{-6})$. We solve

$$\max_y f(0.992533, -1.84104) - f(0.992533, y)$$

and we obtain $PB = (-1.84103, -1.6 \times 10^{-9})$. We set

$$Y_k = \{0, -1.83997, -1.89333, -1.86321, -1.84685, -1.84323, -1.84209, -1.84103\}.$$

We have

$$F(0.992533, -1.84103) < f_0$$

and $f_0 = -1.8125$ is achievable.

Continuing in the same manner we find that the optimum is given at $x = 1$, $y = -1.8401$ with an outer objective value equal to -1.8401.

References

1. Winterfeld, A.: Application of general semi infinite programming to lapidary cutting problems. Technical Report ISSN 1434-9973, Fraunhofer-Institut Techno-and Wirtschaftsmathematik (2006)
2. Hoffman, A., Reinhardt, R.: On reverse chebyshev approximation problems. Technical Report M08/94, Technical University of Illmenau (1994)
3. Kaplan, A., Tichatschke, R.: On a class of terminal variational problems. In: Parametric Optimization and Related Topics, IV (Enschede, 1995) (vol. 9 of Approx. Optim., pp. 185–199). Lang, Frankfurt am Main (1997)
4. Still, G.: Generalized semi-infinite programming: theory and methods. Eur. J. Oper. Res. **119**, 301–313 (1999)
5. Hettich, R., Kortanek, K.O.: Semi-infinite programming: theory, methods, and applications. SIAM Rev. **35**(3), 380–429 (1993)
6. Bhattacharjee, B., Lemonidis, P., Green, W.H. Jr., Barton, P.I.: Global solution of semi-infinite programs. Math. Program **103**(2, Ser. B), 283–307 (2005)
7. Bansal, V., Perkins, J.D., Pistikopoulos, E.N.: Flexibility analysis and design of linear systems by parametric programming. AIChE J. **46**(2), 335–354 (2000)
8. Polak, E., Wets, R.J.-B., Der Kiureghian, A.: On an approach to optimization problems with a probabilistic cost and or constraints. In: Nonlinear Optimization and Related Topics (Erice, 1998) (vol. 36 of Appl. Optim., pp. 299–315). Kluwer Academic Publisher, Dordrecht (2000)
9. Rustem, B., Howe, M.: Algorithms for Worst-Case Design and Applications to Risk Management. Princeton University Press, Princeton, NJ (2002)
10. Kawaguchi, T., Maruyama, Y.: Generalized constrained games in farm planning. Am. J. Agric. Econ. **54**, 591–602 (1972)
11. Lanckriet, G., Ghaoui, L., Bhattacharyya, C., Jordan, M.: A robust minimax approach to classification. J. Mach. Learn. Res. **3**, 555–582 (2002)
12. Hurtado, F., Sacristán, V., Toussaint, G.: Some constrained minimax and maximin location problems. Stud. Locat. Anal. 15, 17–35 (2000)
13. Kiwiel, K.C.: A direct method of linearization for continuous minimax problems. J. Optim. Theory Appl. **55**(2), 271–287 (1987)
14. Monahan, G.E.: Finding saddle points on polyhedra: solving certain continuous minimax problems. Naval Res. Logist. **43**(6), 821–837 (1996)
15. Rustem, B., Zakovic, S., Parpas, P.: Convergence of an interior point algorithm for continuous minimax. J. Optim. Theory Appl. **136**, 87–103 (2008)
16. Sasai, H.: An interior penalty method for minimax problems with constraints. SIAM J. Control **12**, 643–649 (1974)
17. Parpas, P., Rustem, B.: An algorithm for the global optimization of a class of continuous minimax problems. To appear in J. Optim. Theory Appl.
18. Shimizu, K., Aiyoshi, E.: Necessary conditions for min-max problems and algorithms by a relaxation procedure. IEEE Trans. Automat. Control **25**(1), 62–66 (1980)
19. Žaković, S., Rustem, B.: Semi-infinite programming and applications to minimax problems. Ann. Oper. Res. **124**, 81–110 (2003). (Applied mathematical programming and modelling)
20. Royset, J.O., Polak, E., Der Kiureghian, A.: Adaptive approximations and exact penalization for the solution of generalized semi-infinite min-max problems. SIAM J. Optim. **14**(1), 1–33 (electronic) (2003)
21. Visweswaran, V., Floudas, C.A., Ierapetritou, M.G., Pistikopoulos, E.N.: A decomposition-based global optimization approach for solving bilevel linear and quadratic programs. In: State of the Art in Global Optimization (Princeton, NJ, 1995) (vol. 7 of Nonconvex Optim. Appl., pp. 139–162). Kluwer Academic Publisher, Dordrecht (1996)
22. Dua, V., Papalexandri, K.P., Pistikopoulos, E.N.: Global optimization issues in multiparametric continuous and mixed-integer optimization problems. J. Global Optim. **30**(1), 59–89 (2004)
23. Faisca, N., Dua, V., Rustem, B., Saraiva, P., Pistikopoulos, E.: Parametric global optimization for bilevel programming. J. Global Optim. **38**, 609–623 (2007)
24. Mitsos, A., Lemonidis, P., Barton, P.: Global solution of bilevel programs with a nonconvex inner program. J. Global Optim. doi:10.1007/s10898-007-9260-z
25. Stein, O., Still, G.: On generalized semi-infinite optimization and bilevel optimization. Eur. J. Oper. Res. **142**(3), 444–462 (2002)
26. Blankenship, J.W., Falk, J.E.: Infinitely constrained optimization problems. J. Optim. Theory Appl. **19**(2), 261–281 (1976)
27. Polak, E., Royset, J.O., Womersley, R.S.: Algorithms with adaptive smoothing for finite minimax problems. J. Optim. Theory Appl. **119**(3), 459–484 (2003)

28. Polak, E., Royset, J.O.: Algorithms for finite and semi-infinite min-max-min problems using adaptive smoothing techniques. J. Optim. Theory Appl. **119**(3), 421–457 (2003)
29. Tsoukalas, A., Parpas, P., Rustem, B.: A smooting algorithm for min-max-min problems. To be published in Optim. Lett.
30. Parpas, P., Rustem, B., Pistikopoulos, E.N.: Linearly constrained global optimization and stochastic differential equations. J. Global Optim. **36**(2), 191–217 (2006)
31. Horst, R., Tuy, H.: Global Optimization, 2nd edn. Springer-Verlag, Berlin (1993). (Deterministic approaches)